Journal of Engineering Mathematics, Vol. 12, No. 2, April 1978
Sijthoff & Noordhoff International Publishers–Alphen aan den Rijn
Printed in The Netherlands

A general procedure for deriving solutions of dual integral equations

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(Received May 24, 1977)

SUMMARY

In this paper there is developed an operational procedure for deriving solutions of dual integral equations of a general type – equations (1.1) and (1.2) below. The method depends strongly on properties of the Mellin transform. To illustrate the application of the method solutions are derived of the elementary type of dual integral equations occurring in engineering applications. The method is then applied to the solution of dual integral equations of Titchmarsh type and to those involving Y- and K-transforms.

1. Introduction

Dual integral equations of the type

$$\int_0^\infty h_1(xt)\phi(t)dt = f(x), \qquad 0 \leq x < 1, \tag{1.1}$$

$$\int_0^\infty h_2(xt)\phi(t)dt = g(x), \qquad x > 1. \tag{1.2}$$

have been studied extensively and with various degrees of generality particularly when the kernels h_1 and h_2 involve the circular functions or the Bessel functions of the first kind. Many methods have been used to solve equations of this type in the hundred years since they were first considered by Weber. An account of these different methods is given in the introduction to the paper [1] and, at greater length, in Chapter IV of the book [2].

In this paper we first outline (in Sec. 2) a general technique for deriving solutions of equations of the general type (1.1) and (1.2). The method consists in exploiting the properties of the Mellin transform to reduce the problem to that of solving an integral equation of the first kind. A somewhat similar technique has been employed by Williams [3] and Tanno [4]. As in almost all of the papers concerned with the solution of dual integral equations, the analysis is purely formal. In other words we are not endeavouring to establish

* Research supported by a National Research Council of Canada grant.

solutions in a rigorous fashion, but merely to provide "candidates" for such a rigorous analysis. Our approach is the "operational approach" of the applied mathematician; any solutions found by a method of this kind must be subjected to the test of being shown to satisfy the original equations.

In Sec. 3 we apply the method to obtain the solutions of some of the dual integral equations which arise in some of the simpler mixed boundary value problems in electrostatics, elasticity and diffusion theory. The solutions are already known and the purpose of this section is merely to show the technique at work. In Sec. 4 we consider the problem of deriving the solution of Titchmarsh-type equations and in Secs. 5, 6 those of deriving solutions of dual integral equations whose kernels involve, respectively, Y- and K-functions. In using results concerning the Mellin transform we make use of Chapter 4 of [5].

2. Description of the method

The dual integral equations (1.1) and (1.2) are equivalent to the equations

$$\int_0^t m_1(t/x)x^{-1}dx \int_0^\infty h_1(xu)\phi(u)du = \psi_1(t), \qquad 0 < t < 1,$$

$$\int_t^\infty m_2(t/x)x^{-1}dx \int_0^\infty h_2(xu)\phi(u)du = \psi_2(t), \qquad t > 1,$$

where the functions ψ_1 and ψ_2 are defined in terms of the known functions f, g and the (as yet) arbitrary functions m_1, m_2 by the equations

$$\psi_1(t) = \int_0^t f(x)m_1(t/x)x^{-1}dz, \qquad 0 < t < 1, \tag{2.1}$$

$$\psi_2(t) = \int_t^\infty g(x)m_2(t/x)x^{-1}dx, \qquad t > 1. \tag{2.2}$$

Suppose now that we can find functions m_1 and m_2 such that

$$\int_1^\infty m_1(y)h_1(z/y)y^{-1}dy = \int_0^1 m_2(y)h_2(z/y)y^{-1}dy = k(z) \tag{2.3}$$

then the unknown function ϕ is the solution of the integral equation

$$\int_0^\infty \phi(u)k(ut)du = \psi(t) \tag{2.4}$$

where the function ψ is defined in terms of the functions ψ_1 and ψ_2 through the equation

$$\psi(t) = \psi_1(t)H(1-t) + \psi_2(t)H(t-1), \tag{2.5}$$

H denoting the Heaviside unit function.

If we denote the Mellin transform of a function f by $f^* \equiv \mathcal{M}f$, we see that we can write equations (2.3) and (2.4) in the equivalent forms

$$m_1^*(s)h_1^*(s) = m_2^*(s)h_2^*(s) = k^*(s) \tag{2.6}$$

and

$$\phi^*(1-s)k^*(s) = \psi^*(s) \tag{2.7}$$

respectively, where m_1^*, m_2^* are defined by the equations

$$m_1^*(s) = \mathcal{M}[m_1(t)H(t-1); s], \quad m_2^*(s) = \mathcal{M}[m_2(t)H(1-t); s]. \tag{2.8}$$

Alternatively we can write the unknown function in the form

$$\phi(u) = -\Phi'(u) \tag{2.9}$$

where Φ is the solution of the integral equation

$$\int_0^\infty \Phi(u)K(ut)du = t^{-1}\psi(t) \tag{2.10}$$

whose kernel K is defined by $K = k'$ so that

$$K^*(s) = (1-s)k^*(s-1). \tag{2.11}$$

Again by making use of elementary properties of the Mellin transform we can derive alternative forms for the component functions ψ_1 and ψ_2 occurring in equation (2.5). It is easily shown that if there exists a function m_3 such that

$$\mathcal{M}[m_3(t)H(t-1); s] = -s^{-1}m_1^*(s+2) \tag{2.12}$$

then ψ_1 can be expressed in the alternative form

$$\psi_1(t) = \frac{1}{t}\frac{d}{dt}\int_0^t xf(x)m_3(t/x)dx. \tag{2.13}$$

Similarly, if there exists a function m_4 such that

$$\mathcal{M}[m_4(t)H(1-t); s] = s^{-1}m_2^*(s+2) \tag{2.14}$$

then ψ_2 can be expressed by the formula

$$\psi_2(t) = -\frac{1}{t}\frac{d}{dt}\int_t^\infty xg(x)m_4(t/x)dx. \tag{2.15}$$

3. Some simple pairs of dual integral equations

We begin by applying the method to certain simple pairs of dual integral equations which arise frequently in applications.

(i) If $h_1(x) = \cos x$, $h_2(x) = \sin x$, then

$$h_1^*(s) = \Gamma(s)\cos(\tfrac{1}{2}\pi s), \qquad h_2^*(s) = \Gamma(s)\sin(\tfrac{1}{2}\pi s) \tag{3.1}$$

(Cf. pp. 317, 319 of Vol. 1 of [6]) so that, in this case, equation (2.5) reduces to

$$\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s+\frac{1}{2})}\, m_1^*(s) = \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}\, m_2^*(s).$$

We may therefore take

$$m_1^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2\Gamma(1-\frac{1}{2}s)}, \qquad m_2^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}s)}{2\Gamma(\frac{1}{2}+\frac{1}{2}s)}, \tag{3.2}$$

i.e. we may take

$$m_1(t) = (t^2-1)^{-\frac{1}{2}}H(t-1), \qquad m_2(t) = (1-t^2)^{-\frac{1}{2}}H(1-t). \tag{3.3}$$

Further, from equations (2.6), (3.1) and (3.2) we deduce that the function k has Mellin transform k^* defined by the equation

$$k^*(s) = \tfrac{1}{2}\pi\,\frac{2^{s-1}\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}$$

and hence that

$$k(t) = \tfrac{1}{2}\pi J_0(t). \tag{3.4}$$

Inserting this expression into equation (2.4) and solving the resulting integral equation by means of the Hankel inversion theorem (Cf. [5], p. 309) we see that the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(t)\cos(xt)dt = f(x), && 0 < x < 1,\\ &\int_0^\infty \phi(t)\sin(xt)dt = g(x), && x > 1, \end{aligned}\right\} \tag{3.5}$$

has solution

$$\phi(t) = \frac{2t}{\pi}\int_0^1 uJ_0(ut)dt \int_0^u \frac{f(x)dx}{\sqrt{(u^2-x^2)}} + \frac{2t}{\pi}\int_1^\infty uJ_0(ut)du \int_u^\infty \frac{g(x)dx}{\sqrt{(x^2-u^2)}} \tag{3.6}$$

(ii) On the other hand if $h_1(x) = \sin x$, $h_2(x) = \cos x$, equation (2.5) reduces to

$$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\, m_1^*(s) = \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\, m_2^*(s)$$

which may be rewritten as

$$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}s)}\, m_1^*(s) = -\frac{\Gamma(-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\, m_2^*(s).$$

From this equation we see that we may take

$$m_1^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2}s)}{2\Gamma(\frac{1}{2}-\frac{1}{2}s)}, \qquad m_2^*(s) = -\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}s)}{2\Gamma(1+\frac{1}{2}s)}$$

i.e. we may take

$$m_1(t) = t(t^2-1)^{-\frac{1}{2}} H(t-1), \qquad m_2(t) = t(1-t^2)^{-\frac{1}{2}} H(1-t).$$

With this choice of m_1 and m_2 we deduce from equation (2.6) that the corresponding kernel k has Mellin transform k^* defined by the equation

$$k^*(s) = -\frac{2^{s-1}\Gamma(\frac{1}{2}+\frac{1}{2}s)\pi}{s\Gamma(\frac{1}{2}-\frac{1}{2}s)}.$$

From equation (2.10) we deduce that

$$K^*(s) = \tfrac{1}{2}\pi \cdot \frac{2^{s-1}\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}$$

and hence that

$$K(t) = \tfrac{1}{2}\pi J_0(t).$$

In other words we have shown that the solution of the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(t)\sin(xt)dt = f(x), && 0 \le x < 1,\\ &\int_0^\infty \phi(t)\cos(xt)dt = g(x), && x > 1, \end{aligned}\right\} \tag{3.7}$$

may be written in the form

$$\phi(t) = -\frac{2}{\pi}\frac{d}{dt}t\left[\int_0^1 J_0(ut)du \int_0^u \frac{f(x)dx}{\sqrt{(u^2-x^2)}} + \int_1^\infty J_0(ut)du \int_u^\infty \frac{g(x)dx}{\sqrt{(x^2-u^2)}}\right]. \tag{3.8}$$

(iii) If $h_1(x) = J_0(x)$, $h_2(x) = J_1(x)$, then

$$h_1^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}, \quad h_2^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(\frac{3}{2}-\frac{1}{2}s)}$$

so that we may take

$$m_1^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(1-\frac{1}{2}s)}{2\Gamma(\frac{3}{2}-\frac{1}{2}s)}, \quad m_2^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}s)}{2\Gamma(\frac{1}{2}+\frac{1}{2}s)}$$

which is equivalent to taking

$$m_1(t) = (t^2-1)^{-\frac{1}{2}}H(t-1), \quad m_2(t) = (1-t^2)^{-\frac{1}{2}}H(1-t).$$

The corresponding kernel has Mellin transform

$$k^*(s) = 2^{s-2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}s)}{\Gamma(\frac{3}{2}-\frac{1}{2}s)} = \frac{\pi}{\Gamma(\frac{1}{2}s-\frac{1}{2})\Gamma(\frac{3}{2}-\frac{1}{2}s)}\cdot\frac{2^{s-2}\Gamma(\frac{1}{2}s-\frac{1}{2})\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2})}$$
$$= \Gamma(s-1)\sin\{\tfrac{1}{2}(s-1)\pi\}$$

so that

$$k(t) = t^{-1}\sin t \tag{3.9}$$

and the corresponding integral equation can be solved by means of the inversion theorem for the Fourier sine transform. In this way we find that the solution of the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(t)J_0(xt)dt = f(x), \qquad 0 \le x < 1, \\ &\int_0^\infty \phi(t)J_1(xt)dt = g(x), \qquad x > 1, \end{aligned}\right\} \tag{3.10}$$

can be written in the form

$$\phi(t) = \frac{2t}{\pi}\int_0^1 u\sin(ut)du\int_0^u \frac{f(x)}{\sqrt{(u^2-x^2)}}dx + \frac{2t}{\pi}\int_1^\infty u\sin(ut)du\int_u^\infty \frac{g(x)dx}{\sqrt{(x^2-u^2)}}. \tag{3.11}$$

(iv) If $h_1(x) = 2x^{-1}J_0(x)$, $h_2(x) = J_0(x)$, then

$$h_1^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{\Gamma(\frac{3}{2}-\frac{1}{2}s)}, \quad h_2^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}$$

and we may take

$$m_1^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-\frac{1}{2}s)}{2\Gamma(1-\frac{1}{2}s)}, \quad m_2^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}s-\frac{1}{2})}{2\Gamma(\frac{1}{2}s)}.$$

Since

$$\mathscr{M}^{-1}[-s^{-1}m^*(s+2);t] = \tfrac{1}{2}(t^2-1)^{-\frac{1}{2}}H(t-1)$$

it follows from equations (2.12) and (2.13) that we may write

$$\psi_1(t) = \frac{1}{2t}\frac{d}{dt}\int_0^t \frac{x^2 f(x)dx}{\sqrt{(t^2-x^2)}}, \qquad 0 < t < 1.$$

Also

$$m_2(t) = t^{-1}(1-t^2)^{-\frac{1}{2}}H(1-t)$$

so that

$$\psi_1(t) = \frac{1}{t}\int_t^\infty \frac{xg(x)dx}{\sqrt{(x^2-t^2)}}, \qquad t > 1.$$

Further, from equations (2.6), (3.9) and (3.10) we deduce that the kernel of the relevant integral equation has Mellin transform

$$k^*(s) = 2^{s-2}\frac{\Gamma(\frac{1}{2}s-\frac{1}{2})}{\Gamma(1-\frac{1}{2}s)} = \Gamma(s-1)\sin(\tfrac{1}{2}s\pi)$$

from which we deduce that

$$k(t) = t^{-1}\cos t. \tag{3.12}$$

The corresponding integral equation can be solved by means of the inversion theorem for the Fourier cosine transform to give

$$\phi(t) = \frac{2t}{\pi}\int_0^\infty u\psi(u)\cos(ut)du.$$

Making the substitution $f(x) = 2x^{-1}f_1(x)$ in these equations we see that the solution of the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty t^{-1}\phi(t)J_0(xt)dt = f_1(x), \qquad 0 \le x < 1,\\ &\int_0^\infty \phi(t)J_0(xt)dt = g_2(x), \qquad x > 1, \end{aligned}\right\} \tag{3.13}$$

may be written in the form

$$\phi(t) = \frac{2t}{\pi}\int_0^1 \cos(ut)du\frac{d}{du}\int_0^u \frac{xf_1(x)dx}{\sqrt{(u^2-x^2)}} + \frac{2t}{\pi}\int_1^\infty \cos(ut)du\int_u^\infty \frac{xg_2(x)dx}{\sqrt{(x^2-u^2)}}. \tag{3.14}$$

(v) If $h_1(x) = J_0(x)$, $h_2(x) = 2x^{-1}J_0(x)$, the roles of h_1 and h_2 are reversed and we may take

$$m_1^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(1-\frac{1}{2}s)}{2\Gamma(\frac{1}{2}-\frac{1}{2}s)}, \quad m_2^*(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}s)}{2\Gamma(\frac{1}{2}s-\frac{1}{2})}$$

i.e. we may take

$$m_1(t) = (t^2-1)^{-\frac{1}{2}}t^{-1}H(t-1), \quad m_4(t) = \tfrac{1}{2}(1-t^2)^{-\frac{1}{2}}H(1-t)$$

and hence

$$\psi_1(t) = \frac{1}{t}\int_0^t \frac{xf(x)dx}{\sqrt{(t^2-x^2)}}, \qquad 0 \le t < 1$$

$$\psi_2(t) = -\frac{1}{2t}\frac{d}{dt}\int_t^\infty \frac{x^2g(x)dx}{\sqrt{(x^2-t^2)}}, \qquad t > 1.$$

The kernel of the corresponding integral equation turns out to be given by equation (3.9). Solving the integral equation, again by means of the inversion theorem for the Fourier sine transform and replacing $g(x)$ by $2x^{-1}g_2(x)$ we find that the solution of the dual integral equations

$$\begin{aligned} &\int_0^\infty \phi(t)J_0(xt)dt = f_1(x), \qquad 0 < x < 1, \\ &\int_0^\infty t^{-1}\phi(t)J_0(xt)dt = g_2(x), \qquad x > 1, \end{aligned} \tag{3.15}$$

is given by the equation

$$\begin{aligned} \phi(t) = {} & \frac{2t}{\pi}\int_0^1 \sin(ut)du \int_0^u \frac{xf_1(x)dx}{\sqrt{(u^2-x^2)}} \\ & - \frac{2t}{\pi}\int_0^1 \sin(ut)du \frac{d}{du}\int_u^\infty \frac{xg_2(x)dx}{\sqrt{(x^2-u^2)}}. \end{aligned} \tag{3.16}$$

4. Dual integral equations of Titchmarsh type

We now consider the case in which

$$h_1(x) = 2^{2\alpha}x^{-2\alpha}J_\nu(x), \quad h_2(x) = 2^{2\beta}x^{-2\beta}J_\mu(x). \tag{4.1}$$

Here we have

$$h_1^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}\nu-\alpha+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}\nu+\alpha-\frac{1}{2}s)}, \quad h_2^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}\mu-\beta+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}\mu+\beta-\frac{1}{2}s)}. \tag{4.2}$$

In discussing this and subsequent cases it is convenient to introduce the Erdelyi–Kober operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ defined for $\alpha > 0$ by the equations

$$I_{\eta,\alpha}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2 - t^2)^{\alpha-1} t^{2\eta+1} f(t)dt,$$

$$K_{\eta,\alpha}f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-2\alpha-2\eta+1} f(t)dt$$

and for $\alpha < 0$ by the equations

$$I_{\eta,\alpha}f(x) = x^{-2\eta-2\alpha}\left(\frac{1}{2x}\frac{d}{dx}\right)^n x^{2n+2\alpha+2\eta} I_{\eta,\alpha+n}f(x),$$

$$K_{\eta,\alpha}f(x) = (-1)^n x^{2\eta}\left(\frac{1}{2x}\frac{d}{dx}\right)^n x^{2n-2\eta} K_{\eta-n,\alpha+n}f(x)$$

in which equations n is a positive integer such that $0 < \alpha + n < 1$ (Cf. pp. 54–55 of [2]). It is easily shown that

$$\mathcal{M}I_{\eta,\alpha}f(s) = \frac{\Gamma(1 + \eta - \frac{1}{2}s)}{\Gamma(1 + \eta + \alpha - \frac{1}{2}s)} f^*(s), \tag{4.3}$$

$$\mathcal{M}K_{\eta,\alpha}f(s) = \frac{\Gamma(\eta + \frac{1}{2}s)}{\Gamma(\eta + \alpha + \frac{1}{2}s)} f^*(s). \tag{4.4}$$

It is also convenient to use $S_{\eta,\alpha}$, the operator of the modified Hankel transform, defined by the equation

$$S_{\eta,\alpha}f(x) = 2^\alpha x^{-\alpha} \int_0^\infty t^{1-\alpha} f(t) J_{2\eta+\alpha}(xt)dt \tag{4.5}$$

from which we deduce immediately that

$$\mathcal{M}S_{\eta,\alpha}f(s) = 2^{s-1} \frac{\Gamma(\eta + \frac{1}{2}s)}{\Gamma(1 + \eta + \alpha - \frac{1}{2}s)} f^*(2 - s). \tag{4.6}$$

In equations (4.3), (4.4) and (4.6) f^* denotes, as usual $\mathcal{M}f$.

In the sequel we shall make use of the inversion formulae

$$I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha}; \quad K^{-1}_{\eta,\alpha} = K_{\eta+\alpha,-\alpha}; \quad S^{-1}_{\eta,\alpha} = S_{\eta+\alpha,-\alpha}. \tag{4.7}$$

For the forms (4.2) for h_1^*, h_2^* we may take

$$m_1^*(s) = \frac{\Gamma(1 + \frac{1}{2}\nu + \alpha - \frac{1}{2}s)}{\Gamma(1 + \frac{1}{2}\mu + \beta - \frac{1}{2}s)}, \quad m_2^*(s) = \frac{\Gamma(\frac{1}{2}\nu - \alpha + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\mu - \beta + \frac{1}{2}s)}$$

from which, by using equations (4.3) and (4.4) we deduce that

$$\psi_1(t) = I_{\frac{1}{2}\nu+\alpha,\frac{1}{2}\mu-\frac{1}{2}\nu-\alpha+\beta}f(t), \qquad 0 \le t < 1, \tag{4.8}$$

$$\psi_2(t) = K_{\frac{1}{2}\nu-\alpha,\frac{1}{2}\mu-\frac{1}{2}\nu+\alpha-\beta}g(t), \qquad t > 1. \tag{4.9}$$

The corresponding expression for the Mellin transform of the kernel is

$$k^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}\nu-\alpha+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}\mu+\beta-\frac{1}{2}s)}.$$

Making use of equation (4.6) we see that the integral equation for ϕ assumes the form

$$S_{\frac{1}{2}\nu-\alpha,\frac{1}{2}\mu-\frac{1}{2}\nu+\alpha+\beta}\{u^{-1}\phi(u);t\} = \psi(t).$$

Using the third equation of the set (4.7) we deduce that

$$\phi(t) = tS_{\frac{1}{2}\mu+\beta,-\frac{1}{2}\mu+\frac{1}{2}\nu-\alpha-\beta}\psi(t).$$

From the definition (4.5) we see that the solution can be written in the form

$$\phi(t) = 2^{-\gamma}t^{1+\gamma}\int_0^\infty u^{1+\gamma}\psi(u)J_\lambda(tu)du \tag{4.10}$$

with the parameters γ, λ defined by the equations

$$\gamma = \tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \alpha + \beta, \qquad \lambda = \tfrac{1}{2}\mu + \tfrac{1}{2}\nu - \alpha + \beta \tag{4.11}$$

and the function ψ determined by the equations (2.5), (4.8) and (4.9).

Replacing $f(x)$ by $2^{2\alpha}x^{-2\alpha}f_1(x)$ and $g(x)$ by $2^{2\beta}x^{-2\beta}g_2(x)$ we find that

$$\psi_1(t) = 2^{2\alpha}t^{-2\alpha}I_{\frac{1}{2}\nu,\gamma-2\alpha}f_1(t), \qquad 0 \le t < 1, \tag{4.12}$$

$$\psi_2(t) = 2^{2\beta}t^{-2\beta}K_{\lambda-\frac{1}{2}\mu,\gamma-2\beta}g_2(t), \qquad t > 1. \tag{4.13}$$

and hence that the solution of the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty t^{-2\alpha}\phi(t)J_\nu(xt)dt = f_1(x), \qquad 0 \le x < 1,\\ &\int_0^\infty t^{-2\beta}\phi(t)J_\mu(xt)dt = g_2(x), \qquad x > 1, \end{aligned}\right\} \tag{4.14}$$

is given by equation (4.10) with the function ψ defined by equations (2.5), (4.12) and (4.13).

For example if

$$\mu - \nu > 2|\alpha - \beta|$$

the solution can be written in the form

$$\phi(t) = \frac{2^{2\alpha-\gamma+1}}{\Gamma(\gamma-2\alpha)} t^{1+\gamma} \int_0^1 u^{1-\lambda} J_\lambda(tu)du \int_0^u (u^2-x^2)^{\gamma-2\alpha-1} x^{\nu+1} f_1(x)dx$$
$$+ \frac{2^{2\beta-\gamma+1}}{\Gamma(\gamma-2\beta)} t^{1+\gamma} \int_1^\infty u^{1+\lambda} J_\lambda(tu)du \int_u^\infty (x^2-u^2)^{\gamma-2\beta-1} x^{-\mu+1} g_2(x)dx. \quad (4.15)$$

5. Dual integral equations involving Y-functions

In the case in which $h_1(x) = Y_\nu(x)$, $h_2(x) = Y_\mu(x)$, we have

$$h_1^*(s) = \frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)\Gamma(\frac{1}{2}s-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}s-\frac{1}{2}\nu-\frac{1}{2})\Gamma(\frac{3}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}, \quad h_2^*(s) = \frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\mu)\Gamma(\frac{1}{2}s-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}s-\frac{1}{2}\mu-\frac{1}{2})\Gamma(\frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}s)}$$

(Cf. p. 329 of Vol. I of [6]) so that we may take

$$m_1^*(s) = \frac{\Gamma(\frac{3}{2}+\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}s)}, \quad m_2^*(s) = \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)\Gamma(\frac{1}{2}s-\frac{1}{2}\nu)\Gamma(\frac{1}{2}s-\frac{1}{2}\mu-\frac{1}{2})}{\Gamma(\frac{1}{2}s+\frac{1}{2}\mu)\Gamma(\frac{1}{2}s-\frac{1}{2}\mu)\Gamma(\frac{1}{2}s-\frac{1}{2}\nu-\frac{1}{2})}.$$

Hence we deduce that

$$\psi_1(t) = I_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\mu-\frac{1}{2}\nu} f(t), \qquad 0 \le t < 1, \quad (5.1)$$

$$\psi_2(t) = t^{-\mu-\nu-1} K_{\frac{1}{2}\mu+\frac{1}{2},-\frac{1}{2}\mu+\frac{1}{2}\nu} K_{\frac{1}{2}\mu+\nu+\frac{1}{2},\frac{1}{2}\mu-\frac{1}{2}\nu} K_{\frac{1}{2}\nu,\frac{1}{2}\mu-\frac{1}{2}\nu}\{x^{\mu+\nu+1}g(x);t\}. \quad (5.2)$$

Further,

$$k^*(s) = \frac{2^{s-1}\Gamma(\frac{1}{2}s+\frac{1}{2}\nu)\Gamma(\frac{1}{2}s-\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}s-\frac{1}{2}\nu-\frac{1}{2})\Gamma(\frac{3}{2}+\frac{1}{2}\mu-\frac{1}{2}s)}.$$

It is easily shown that

$$\mathscr{M}[S_{-\frac{1}{2}\nu,\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}} I_{\frac{1}{2}\nu,-\nu-\frac{1}{2}}\{x^{-1}\phi(x);t\};s] = k^*(s)\phi^*(1-s)$$

and hence that ϕ is the solution of the integral equation

$$S_{-\frac{1}{2}\nu,\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}} I_{\frac{1}{2}\nu,-\nu-\frac{1}{2}}\{x^{-1}\phi(x);t\} = \psi(t).$$

Making use of the formulae for the inverses of the I- and S-operators we deduce from this last equation that the solution of the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(t)Y_\nu(xt)dt = f(x), \qquad 0 \le x < 1, \\ &\int_0^\infty \phi(t)Y_\mu(xt)dt = g(x), \qquad x > 1, \end{aligned}\right\} \quad (5.3)$$

may be written in the form

$$\phi(t) = tI_{-\frac{1}{2}\nu-\frac{1}{2},\,\nu+\frac{1}{2}}S_{\frac{1}{2}\mu+\frac{1}{2},\,-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}}\psi(t) \tag{5.4}$$

where $\psi(t) = \psi_1(t)H(1-t) + \psi_2(t)H(t-1)$ is given by equations (5.1) and (5.2).

The solution (4.4) when written out in conventional form is

$$\phi(t) = \frac{2^{\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu}}{\Gamma(\nu+\frac{1}{2})}\int_0^t (t^2-\xi^2)^{\nu-\frac{1}{2}}\xi^{\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}}d\xi \int_0^\infty \tau^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{3}{2}}J_{\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}}(\xi\tau)\psi(\tau)d\tau. \tag{5.5}$$

For example, if we take $\nu = 0$, $\mu = 1$, $f(x) \equiv 1$, $0 < x < 1$, $g(x) \equiv 0$, $x > 1$ we find that $\psi(t) = \frac{1}{2}\sqrt{\pi H(1-t)}$ and hence that

$$\phi(t) = \frac{1}{2}\int_0^t \frac{\xi d\xi}{\sqrt{(t^2-\xi^2)}}\int_0^1 \tau^2 J_1(\xi\tau)d\tau.$$

Recalling the formula

$$\int_0^1 \tau^2 J_1(\xi\tau)d\tau = J_2(\xi)/\xi$$

we see that

$$\phi(t) = \frac{1}{4}\int_0^{t^2}\frac{x^{-\frac{1}{2}}J_2(x^{\frac{1}{2}})dx}{\sqrt{(t^2-x)}}$$

and hence, making use of the formula (59) on p. 194 of Vol. 2 of [6], we deduce that

$$\phi(t) = \frac{1}{4}\pi\{J_1(\tfrac{1}{2}t)\}^2 \tag{5.6}$$

is the solution of the pair of dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(t)Y_0(xt)dt = 1, \qquad 0 < x < 1, \\ &\int_0^\infty \phi(t)Y_1(xt)dt = 0, \qquad\qquad x > 1. \end{aligned}\right\} \tag{5.7}$$

6. Dual integral equations involving *K*-functions

If we take $h_1(x) = K_\nu(x)$, $h_2(x) = K_\mu(x)$, then we have

$$h_1^*(s) = 2^{s-2}\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu), \quad h_2^*(s) = 2^{s-2}\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\mu).$$

(Cf. formula (26) p. 331 of Vol. I of [6]). We may therefore take

$$m_1^*(s) = \frac{2}{s-\mu}, \qquad m_2^*(s) = \frac{\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\mu + 1)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\mu)}$$

from which we deduce immediately that

$$\psi_1(t) = -2t^{-\mu}\int_0^t x^{\mu-1} f(x)dx, \qquad 0 < t < 1, \tag{6.1}$$

$$\psi_2(t) = K_{-\frac{1}{2}\nu, \frac{1}{2}\nu - \frac{1}{2}\mu} K_{\frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu - 1} g(t), \qquad t > 1,\ (-1 < \nu < 1). \tag{6.2}$$

Also

$$k^*(s) = \frac{\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\mu + 1)} 2^{s-2}\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu).$$

Using the results

$$\mathscr{M}[(\tfrac{1}{2}x)^{-\frac{1}{2}\mu - \frac{1}{2}\nu} K_{\frac{1}{2}\mu - \frac{1}{2}\nu}(x); s] = 2^{s-2}\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu),$$

$$\mathscr{M} I_{\frac{1}{2}\nu - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu}\phi(x); 1-s] = \frac{\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2}s + 1 - \tfrac{1}{2}\mu)}\phi^*(1-s)$$

we see that

$$\int_0^\infty k(ut)\phi(u)du = \int_0^\infty (\tfrac{1}{2}ut)^{-\frac{1}{2}\mu - \frac{1}{2}\nu} K_{\frac{1}{2}\mu - \frac{1}{2}\nu}(ut) I_{\frac{1}{2}\nu - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu}\phi(u)du.$$

From this we deduce immediately that the solution of the dual integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(u)K_\nu(xu)du = f(x), \qquad 0 \le x < 1, \\ &\int_0^\infty \phi(u)K_\mu(xu)du = g(x), \qquad x > 1, \end{aligned}\right\} \tag{6.3}$$

$(-1 < \nu < 1, \mu - \nu < 3)$ may be written in the form

$$\phi(u) = u^{\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}} I_{-\frac{1}{4}\mu + \frac{1}{4}\nu + \frac{1}{4}, \frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}}\chi(u) \tag{6.4}$$

where χ is the solution of the integral equation

$$\int_0^\infty (ut)^{\frac{1}{2}} K_{\frac{1}{2}\mu - \frac{1}{2}\nu}(ut)\chi(u)du = \hat{\psi}(t) \tag{6.5}$$

with $\hat{\psi}$ defined by the equation

$$\hat{\psi}(t) = 2^{-\frac{1}{2}\mu - \frac{1}{2}\nu} t^{\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}}\{\psi_1(t)H(1-t) + \psi_2(t)H(t-1)\}, \tag{6.6}$$

ψ_1 and ψ_2 being given by equations (6.1) and (6.2).

The solution of the integral equation (6.5) is well known; it is obtained immediately by applying the inversion formula for the K-transforms introduced by Meijer [7].

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